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Hierarchical Ferromagnetic Vector Spin Model Possessing the Lee–Yang Property. Thermodynamic Limit at the Critical Point and Above

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The hierarchical ferromagnetic N-dimensional vector spin model as a sequence of probability measures on $\mathbf{R}^N$ is considered. The starting element of this sequence is chosen to belong to the Lee–Yang class of measures that is defined in the paper and includes most known examples (φ^4 measures, Gaussian measures, and so on). For this model, we prove two thermodynamic limit theorems. One of them is just the classical central limit theorem for weakly dependent random vectors. It describes the convergence of classically normed sums of spins when temperature is sufficiently high. The other theorem describes the convergence of "more than normally" normed sums that holds for some fixed temperature. It corresponds to the strong dependence of spins, which appears at the critical point of the model.

KEY WORDS: Hierarchical ferromagnetic vector spin model; Gibbs measure; Lee–Yang property; Laplace transformation; critical point; thermodynamic limit theorem.

1. INTRODUCTION AND SETUP

The main idea of this paper is that the subsequent progress of statistical physics will be connected with the extensive employment of methods of analytic functions theory as took place earlier with probability theory. These methods usually are applied to study such objects as probability distributions (and Gibbs measures) on the basis of Fourier-like transformations. The transformation used in this paper is the Laplace one. We exploit it to establish the notion of the "Lee–Yang property," which plays

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a key role in this paper. We prove two theorems, which describe the thermodynamic limit at the critical point and above it (for $T > T_{\rm cr}$), for the hierarchical ferromagnetic N-vector model possessing the Lee–Yang property. The latter means that the probability measure that describes the initial distribution of spins (the Gibbs measure with zero interaction between the spins) possesses this property. Although similar and much more sophisticated results were obtained for the choice of this initial measure of the well-known φ^4 type—first by Bleher[1-3] and then by other authors (see, for example, ref. 5 and references in ref. 3)—our theorems describe the whole class of measures and are proved by means of the Lee–Yang property, which can be considered as an attempt to apply the methods mentioned above. Moreover, it becomes clear that the class of measures possessing the Lee–Yang property forms the natural family of measures for which the "usual" properties are typical. This implies if one wants to obtain some "unusual" properties of the model, one should choose the initial distribution out of the Lee–Yang class of measures. The detailed consideration of the connection between the analytical properties of the Laplace transforms of Gibbs measures and the thermodynamic properties of corresponding models will be done in a forthcoming paper.

This work is a direct continuation of our earlier paper[8] and its main results which we need here are sketched below. Let Ξ be a countable set and for every $n \in \mathbf{Z}_+ \equiv \{m \in \mathbf{Z} \mid m \geqslant 0\}$, let there be given a partition of Ξ on subsets $\Lambda_r^{(n)}$, $r \in \mathbf{Z}_+$, each of which consists of δ (a certain integer number not less than two) subsets $\Lambda^{(n-1)}$. The zero-level subsets are the elements of Ξ themselves. For each $i \in \Xi$, let a random vector $\sigma(i)$ (spin) be given and for some finite $\Lambda \subset \Xi$, we denote $\sigma(\Lambda) = \sum_{i \in \Lambda} \sigma(i)$. Within the framework of the hierarchical model, the spins $\sigma(i)$ are dependent in such a way that the probability distribution of $\sigma(\Lambda^{(n)})$ is defined by that of $\sigma(\Lambda^{(n-1)})$ as follows. For every Borel subset $B \subseteq \mathbf{R}^N$, we denote

$$\operatorname{Prob}\{\sigma(\Lambda^{(n)}) \in B\} = P_n(B); \qquad P_0 = \chi$$

Then

$$\begin{aligned} dP_n(\sigma) &= Y_n^{-1} \exp[\tfrac{1}{2}\beta(\delta^{\lambda} - 1)\, \delta^{-n(1+\lambda)} \sigma^2]\, T_n(\sigma)\, d\sigma \\ T_n(\sigma) &= \int_{\mathbf{R}^{\delta N}} \delta\left(\sigma - \sum_{r=1}^{\delta} \sigma_r\right) \prod_{r=1}^{\delta} dP_{n-1}(\sigma_r) \end{aligned} \tag{1.1}$$

where $\sigma^2 = (\sigma, \sigma)$ is the scalar product in $\mathbf{R}^N$; $\delta(\cdot)$ is the Dirac δ-function; Y_n is given by the normalization condition $P(\mathbf{R}^N) = 1$; χ is some probability measure describing the initial distribution of $\sigma(i)$; $\beta \geqslant 0$ is the inverse temperature; and $\lambda > 0$ is the parameter which describes the decay of the interaction between the spins. The ferromagnetic nature of our model lies in the fact that the quadratic form of σ in $\exp(\cdot)$ in the expression (1.1), which describes the interaction between the spins in $\Lambda^{(n)}$, is positive.

The natural physical condition imposed on the initial measure χ is the existence of its Laplace transform

$$F_\chi(v) = \int_{\mathbf{R}^N} \exp(v, \sigma)\, d\chi(\sigma), \qquad v \in \mathbf{R}^N \tag{1.2}$$

as a suitable function defined on $\mathbf{R}^N$. If this condition is fulfilled, all P_n possess this property, provided they exist as measures. Let $O(N)$ be the group of orthogonal transformations of $\mathbf{R}^N$. For $B \subset \mathbf{R}^N$ and $U \in O(N)$, we set $UB = \{v \in \mathbf{R}^N \mid U^{-1} v \in B\}$.

Definition 1.1. The measure χ is said to be isotropic if for any Borel subset B and arbitrary $U \in O(N)$, $\chi(B) = \chi(UB)$.

Let $\mathscr{L}$ denote the family of entire functions on $\mathbf{C}$ that can be written

$$\begin{gathered} f(z) = \exp(\theta z) \prod_{j=1}^{\infty} (1 + \gamma_j z) \\ \theta \geqslant 0; \qquad \gamma_j \geqslant \gamma_{j+1} \geqslant 0; \qquad \sum_j \gamma_j < \infty \end{gathered} \tag{1.3}$$

Definition 1.2. The measure χ is said to possess the Lee–Yang property if there exists $f_\chi(z) \in \mathscr{L}$ such that $F_\chi(v)$ defined by (1.2) satisfies the relation $F_\chi(v) = f_\chi(v^2)$ for all $v \in \mathbf{R}^N$. The set of all measures possessing the Lee–Yang property is denoted $\mathscr{M}_N$.

The detailed description of such measures can be found in ref. 9. Here we only point out some facts which are used in our consideration. The isotropic Gaussian measures possess the Lee–Yang property and for each such measure μ the corresponding function f_μ is $\exp(\theta z)$ with $\theta = \langle \sigma^2 \rangle_\mu / 2N$, where $\langle \cdot \rangle_\mu$ denotes the expectation. Every $f \in \mathscr{L}$ differs from zero in some neighborhood of the point $z = 0$; thus there $\varphi(z) = \log f(z)$ is a holomorphic function possessing the following series expansion representation:

$$\varphi(z) = \sum_{k=1}^{\infty} \frac{1}{k!} \varphi^{(k)} z^k \tag{1.4}$$

Proposition 1.1. The derivatives of φ defined by (1.4) obey the sign rule

$$(-1)^{k-1}\varphi^{(k)}\geqslant 0, \qquad k\in\mathbf{N} \tag{1.5}$$

The equalities in (1.5) are possible simultaneously for all $k\geqslant 2$ and only in the case $f(z)=\exp(\theta z)$.

The proof follows from (1.3). The next assertion was proved in ref. 10.

Proposition 1.2. Let $\chi\in\mathscr{M}_N$ and $f_\chi(z)$ be given by Definition 1.2. Then the derivatives $\varphi_\chi^{(k)}=(D^k\log f_\chi)(0)$, $k=1,2$, obey the estimate

$$\frac{|\varphi_\chi^{(2)}|}{(\varphi_\chi^{(1)})^2}\leqslant\frac{2}{N+2} \tag{1.6}$$

Remark 1.1. For given $f_\chi\in\mathscr{L}$, we define $m_k=\sum_j\gamma_j^k$, $k\in\mathbf{N}$, where γ_j are as in (1.3). Thus $\varphi_\chi^{(1)}=\theta+m_1$, $\varphi_\chi^{(k)}=(-1)^{k-1}(k-1)!\,m_k$ for $k\geqslant 2$, and (1.6) can be rewritten as

$$\frac{m_2}{(\theta+m_1)^2}\leqslant\frac{2}{N+2}$$

Proposition 1.3. For given $\beta>0$ and $n\in\mathbf{Z}_+$, let the measure P_n defined by the relation (1.1) exist and the initial measure χ belong to $\mathscr{M}_N$. Then P_n also belongs to $\mathscr{M}_N$.

This statement is equivalent to Theorem 3.1 proved in ref. 8. It plays a key role in our consideration. For $\beta=0$, P_n is the convolution of an appropriate number of measures P_{n-1} which corresponds to the absence of the dependence between the spins $\sigma(i)$. In this case, the classical central limit theorem for independent identically distributed random vectors[7] ought to hold. In order to obtain the nontrivial limit of the sequence of sums $\{\sigma(\Lambda^{(n)})\}$, it is necessary to normalize them in accordance with this theorem by putting $\sigma(\Lambda^{(n)})\to\sigma(\Lambda^{(n)})\,\delta^{-n/2}$ (recall that $\Lambda^{(n)}$ contains δ^n points). Besides, we can put $\sigma(\Lambda^{(n)})\to\sigma(\Lambda^{(n)})\,\delta^{-n(1+\rho)/2}$ with some $\rho>0$. The latter is known[11] as the "abnormal" normalization is contrast to the case of independent spins. Suppose, for some $\beta>0$, the sequence $\{\sigma(\Lambda^{(n)})\,\delta^{-n/2}\}$ converges to a nontrivial (nondegenerate) limit. Then the weak dependence between the spins[7] holds at this temperature. But if, for some $\beta>0$ and $\rho>0$, one gets that the sequence $\{\sigma(\Lambda^{(n)})\,\delta^{-n(1+\rho)/2}\}$ converges to the nontrivial limit, then the strong dependence between the spins appears. This is peculiar to the critical point of the model. Both such convergences are shown to hold below.

Let us describe the model we deal with more precisely. First, we restrict the values of λ to the interval $(0, 1/2)$. Now let $\bar{\varepsilon} = \min\{1/6; (1-2\lambda)/4\}$ and

$$a(\lambda) = \delta^{1/2} \frac{\delta^{\bar{\varepsilon}} - 1}{\delta^{\lambda + \bar{\varepsilon}} - 1} \tag{1.7}$$

Definition 1.3. For given $\lambda \in (0, 1/2)$, let $\mathcal{M}_N(\lambda)$ be the subset of $\mathcal{M}_N$ consisting of measures χ with the properties

$$0 < \frac{|\varphi_\chi^{(2)}|}{(\varphi_\chi^{(1)})^2} < \frac{2}{N+2} a(\lambda); \qquad \frac{m_2}{m_1^2} \leqslant \frac{2}{N+2} \tag{1.8}$$

where $\varphi_\chi^{(k)}$, m_k are as in Proposition 1.2 and Remark 1.1.

The case of purely Gaussian initial measures (when $\varphi_\chi^{(2)} = 0$) is trivial—all elements of the sequence $\{P_n\}$ can be computed explicitly [see below (3.2) with $h_n = 1$].

For λ close to zero, $a(\lambda)$ is close to $\delta^{1/2}$, which means $\mathcal{M}_N(\lambda)$ includes almost all non-Gaussian measures from $\mathcal{M}_N$ [see (1.6)]. For λ close to 1/2, $\bar{\varepsilon}$ and $a(\lambda)$ are close to zero and the restriction of $|\varphi_\chi^{(2)}|$ used in the definition of $\mathcal{M}_N(\lambda)$ becomes essential; but in any case $\mathcal{M}_N(\lambda)$ is nonempty (see Remark 1.3). Denote

$$dP_n^{(\rho)}(\sigma) = dP_n(\sigma \delta^{(n/2)(1+\rho)}), \qquad n \in \mathbf{Z}_+ \tag{1.9}$$

with P_n given by the recursive formula (1.1).

Theorem 1. Let $\lambda \in (0, 1/2)$ and $\chi \in \mathcal{M}_N(\lambda)$; then there exists $\beta_* > 0$ such that the sequence $\{P_n^{(\lambda)}, n \in \mathbf{Z}_+ \mid P_0^{(\lambda)} = \chi, \beta = \beta_*\}$, defined by (1.1) and (1.9), weakly converges to the isotropic Gaussian measure with the variance N/β_*.

Theorem 2. Let λ, χ, β_* be as above; then the sequence $\{P_n^{(0)}, n \in \mathbf{Z}_+ \mid P_0^{(0)} = \chi, \beta < \beta_*\}$ weakly converges to some isotropic Gaussian measure.

Remark 1.2. The measure $d\chi(\sigma) = C \exp[a(\sigma)^2 - u(\sigma^2)^2]\, d\sigma$ belongs to $\mathcal{M}_N$ for all $N \in \mathbf{N}$, $a \in \mathbf{R}$, $u > 0$.[9] It may belong also to $\mathcal{M}_N(\lambda)$ provided u is sufficiently small [if $a(\lambda)$ in (1.8) is small]. Bleher[1]($N = 1$) and Bleher and Major[2] ($N > 1$) proved similar convergence for the initial measure χ of such type with $a = -1$ and u closed to zero. A detailed survey of the results in this direction can be found in ref. 3. Theorem 4.1 of this paper is related to the matter of this remark.

Remark 1.3. As follows from Definition 1.3, the family $\mathscr{M}_N(\lambda)$ consists of measures χ with sufficiently small $|\varphi_\chi^{(2)}|$. In order to show that $\mathscr{M}_N(\lambda)$ is not empty, let us consider $\chi \in \mathscr{M}_N$ such that f_χ is given by (1.3) with $\theta = 0$. This measure exists.[9,10] Then the second condition (1.8) holds [in view of (1.6)]. If for this measure $|\varphi_\chi^{(2)}|$ is too large and the first restriction in (1.8) does not hold, we construct a new measure by taking the convolution of χ with some Gaussian isotropic measure which has the variance large enough. This new measure has the same $\varphi^{(2)}$, but large enough $\varphi^{(1)}$, and hence belongs to $\mathscr{M}_N(\lambda)$.

Now we use Proposition 1.3. For each P_n, there exist the function $f_{P_n} \in \mathscr{L}$,

$$f_{P_n}(v^2) = \int_{\mathbf{R}^N} \exp(v, \sigma)\, dP_n(\sigma) \tag{1.10}$$

in accordance with Definition 1.2. The relation (1.1) implies that the sequence $\{f_{P_n}(z), n \in \mathbf{Z}_+\}$ can be arranged also recursively as follows:

$$f_{P_n}(z) = \frac{1}{Y_n} \exp\left[\frac{1}{2}\beta(\delta^\lambda - 1)\,\delta^{-n(1+\lambda)}\Delta_N\right][f_{P_{n-1}}(z)]^\delta \tag{1.11}$$

$$Y_n = \left\{\exp\left[\frac{1}{2}\beta(\delta^\lambda - 1)\,\delta^{-n(1+\lambda)}\Delta_N\right][f_{P_{n-1}}(z)]^\delta\right\}_{z=0}$$

$$f_{P_0}(z) = f_\chi(z)$$

$$\Delta_N = 2ND + 4zD^2; \qquad D = \frac{d}{dz} \tag{1.12}$$

The formal operator $\exp(a\Delta_N)$ above will be defined in appropriate topological spaces of entire functions as a continuous operator. Then the uniform convergence on compact subsets of $\mathbf{C}$ of the sequences defined by (1.11) will be shown. This corresponds to the weak convergence mentioned in Theorems 1 and 2 that is established by the continuity theorem (ref. 7, p. 27).

2. MAIN OPERATOR

For some $b > 0$ and an entire function $f(z)$, we define

$$\|f\|_b = \sup\{b^{-k}\,|f^{(k)}| \mid k \in \mathbf{Z}_+\}; \qquad f^{(k)} = (D^{(k)}f)(0)$$

and

$$[\exp(a\Delta_N) f](z) = \sum_{k=0}^{\infty} \frac{a^k}{k!} (\Delta_N^k f)(z) \tag{2.1}$$

Proposition 2.1. For given function $f(z)$ and $b > 0$, let $\|f\|_b < \infty$. Then, for all $a \in (0, 1/4b)$, $[\exp(a\Delta_N) f](z)$ defines an entire function such that

$$\|\exp(a\Delta_N) f\|_c \leqslant l(1-4ab)^{-N/2} \|f\|_b; \qquad c = b(1-4ab)^{-1}$$

The proof follows from Lemma 2.8 of ref. 8. Denote

$$\mathscr{A}_a = \{f(z) \mid \|f\|_b < \infty, \forall b > a\}$$

This set equipped with the pointwise linear operations and the topology generated by the family $\{\|\cdot\|_b, b > a\}$ becomes a Fréchet space with the following properties.[8]

Proposition 2.2. The set of all polynomials is dense in $\mathscr{A}_a$.

Proposition 2.3. The relative topology on every bounded subset $B \subset \mathscr{A}_a$ coincides with that of the uniform convergence on compact subsets of **C**.

For the set $\mathscr{L}$ defined by (1.3), denote $\mathscr{L}_a = \mathscr{L} \cap \mathscr{A}_a$. By means of Proposition 2.1 and the properties of Δ_N defined by (1.12), the following assertion was proved.[8]

Proposition 2.4. For all $n \in \mathbf{N}$, $N \in \mathbf{N}$, and $a \in (0, 1/4b)$, the operator $\exp(a\Delta_N)$ maps $\mathscr{L}_b$ into $\mathscr{L}_c$, where $c = b(1-4ab)^{-1}$.

Therewith, the validity of Proposition 1.3 follows immediately. In the sequel, we will use the following variant of the $\exp(a\Delta_N)$ definition. We set

$$[\exp(a\Delta_N^*) f](z) = \exp\left(-\frac{z}{4a}\right) \int_0^{\infty} s^{N/2-1} e^{-s} w_N\left(\frac{sz}{a}\right) f(4as)\, ds \tag{2.2}$$

where

$$w_N(z) = \sum_{l=0}^{\infty} \frac{z^l}{2^{2l} l!\, \Gamma(N/2+l)} \tag{2.3}$$

By means of the identity

$$\frac{\Gamma(z+M+N)}{\Gamma(z+M)\,\Gamma(z+N)}=\sum_{n=0}^{\min(M,N)} C_M^n C_N^n \frac{n!}{\Gamma(z+n)}; \qquad z\in\mathbf{C};\quad M, N, \in\mathrm{N} \tag{2.4}$$

we can prove the following statement.

Proposition 2.5. For given $b>0$ and $a\in(0, 1/4b)$, both expressions (2.1) and (2.2) define the same operator on $\mathscr{A}_a$.

Proof. Proposition 2.2 implies that this statement needs to be proved only for $f(z)=z^m$. From (2.1)

$$\exp(a\Delta_N)\, z^m=\sum_{k=0}^{m}\frac{a^k}{k!}\frac{2^{2m}m!\,\Gamma(N/2+m)\,z^{m-k}}{2^{2(m-k)}(m-k)!\,\Gamma(N/2+m-k)} \tag{2.5}$$

At the same time, the definition (2.2) yields

$$\exp(a\Delta_N)\, z^m=\exp\left(-\frac{z}{4a}\right)\sum_{k=0}^{\infty}\frac{2^{2m}a^{m-k}z^k}{2^{2k}k!\,\Gamma(N/2+k)}\,\Gamma\left(\frac{N}{2}+m+k\right) \tag{2.6}$$

Taking into account the identity (2.4), we have

$$\exp(a\Delta_N)\, z^m=\exp\left(-\frac{z}{4a}\right)\sum_{k=0}^{\infty}\frac{2^{2m}a^{m-k}z^k}{2^{2k}k!}\,\Gamma\left(\frac{N}{2}+m\right)$$
$$\sum_{n=0}^{\min(m,k)}\frac{k!}{n!\,(k-n)!}\frac{m!}{n!(m-n)!}\frac{n!}{\Gamma(N/2+n)}$$
$$=\exp\left(-\frac{z}{4a}\right)\sum_{n=0}^{m}\frac{z^n m!\,2^{2m}a^{m-n}}{n!\,(m-n)!\,2^{2n}}\frac{\Gamma(N/2+m)}{\Gamma(N/2+n)}\sum_{k=n}^{\infty}\frac{z^{k-n}a^{n-k}}{2^{2(k-n)}(k-n)!}$$
$$=\sum_{n=0}^{m}\frac{z^n m!\,2^{2m}a^{m-n}}{n!\,(m-n)!\,2^{2n}}\frac{\Gamma(N/2+m)}{\Gamma(N/2+n)}=\mathrm{RHS}(2.5)\quad\blacksquare$$

Now, for $t>0$, let us consider the action of $\exp(\frac{1}{2}t\,\Delta_N)$ defined by (2.2) on the function $g(z)=\exp(\frac{1}{2}uz)\,f(z)$ with $f(z)\in\mathscr{L}$. For $u>0$, $g(z)$ belongs to $\mathscr{L}$. Thus the function

$$g_t(z)=[\exp(\tfrac{1}{2}t\,\Delta_N)\,g]\,(z) \tag{2.7}$$

also belongs to $\mathscr{L}$. For $u<0$, the function $g(z)$ may not belong to $\mathscr{L}$, but the situation with $g_t(z)$ given by (2.7) remains under the control due to the integral form of $\exp(\frac{1}{2}t\,\Delta_N)$ and the following identity proved in ref. 8:

$$\exp\left(\frac{1}{2}t\,\Delta_N\right)\exp\left(\frac{1}{2}uz\right)f(z)$$
$$=(1-ut)^{-N/2}\exp\left(\frac{1}{2}\frac{uz}{1-ut}\right)$$
$$\times\exp\left[\frac{1}{2}t(1-ut)\,\Delta_N\right]f\left(\frac{z}{(1-ut)^2}\right);\qquad ut<1 \tag{2.8}$$

Lemma 2.1. For given $g(z)$, let there exist $f(z)\in\mathscr{L}_a$ such that $g(z)=\exp(-\frac{1}{2}uz)\,f(z)$ with some $u>2a$. Then, for all $t\geqslant 0$, (2.7) defines an entire function $g_t(z)$, which can be written

$$g_t(z)=\exp(-\tfrac{1}{2}u_t z)\,f_t(z)$$

with $u_t>0$; $f_t(z)\in\mathscr{L}$.

Proof. Making use of the definition (2.2), we get

$$g_t(z)=\exp\left(-\frac{z}{2t}\right)h_t(z) \tag{2.9}$$

For $f(z)\in\mathscr{L}_a$, all $t>0$, $s>0$, $b>a$,

$$0<f(2ts)\leqslant\|f\|_b\exp(2tsb) \tag{2.10}$$

Consider

$$h_t(z)=\int_0^\infty s^{N/2-1}e^{-s}w_N\left(\frac{2sz}{t}\right)\exp(-tus)\,f(2ts)\,ds;\qquad h_t^{(n)}(0)=(D^n h_t)(0)$$

For $u>2a$, we choose $b\in(a,u/2)$. Then the estimates (2.10) yield

$$0<h_t^{(n)}(0)\leqslant\|f\|_b\,(2t)^{-n}\,[1+t(u-2b)]^{-N/2-n}$$

This implies that both functions $h_t(z)$ and $g_t(z)$ are entire for all $t>0$. By means of (2.8), we get

$$g_t(z)=(1+ut)^{-N/2}\exp\left(-\frac{1}{2}\frac{uz}{1+ut}\right)f_t(z)$$

$$f_t(z)=\exp\left[\frac{1}{2}t(1+ut)\,\Delta_N\right]f\left(\frac{z}{(1+ut)^2}\right)$$

For $f(z) \in \mathscr{L}_a$, $f(z/(1+ut)^2) \in \mathscr{L}_{a(t)}$ with $a(t) = a/(1+ut)^2$. In order to prove the existence of $f_t(z)$ as an entire function, we should show that the following inequality is satisfied for all $t > 0$: $4(t/2)(1+ut)\, a(t) < 1$. But

$$4\left(\frac{1}{2}t\right)(1+ut)\, a(t) = \frac{2at}{1+ut} < \frac{ut}{1+ut} < 1, \qquad ut > 0$$

Thus Proposition 2.5 yield $f_t(z) \in \mathscr{L}_b$ with $b = a(1+ut)^{-1}\,[1 + (u-2a)\,t]^{-1}$. ∎

Below we use the following notations: $\psi(z) = \log g(z)$; $\psi_t(z) = \log g_t(z)$. These functions are holomorphic at the point $z = 0$ whenever $g(z)$ obeys the conditions of Lemma 2.1.

Lemma 2.2. Suppose the conditions of Lemma 2.1 are fulfilled. Then the derivatives $\psi_t^{(k)} = (D^k \psi_t)(0)$ obey the sign rule

$$(-1)^{k-1}\, \psi_t^{(k)} \geqslant 0, \qquad \forall t \geqslant 0, \quad \forall k \geqslant 2 \tag{2.11}$$

The equalities are possible simultaneously for all k and only for $f(z) = \exp(\theta z)$.

The proof follows from Proposition 1.1, Remark 1.1, and Lemma 2.1.

Lemma 2.3. Suppose the conditions of Lemma 2.1 are fulfilled. Then there exists at most one value of $t \geqslant 0$ satisfying the equations $\psi_t^{(1)} = 0$. For this value, $\psi_t^{(1)}$ is strictly decaying as a function of t.

Proof. The function $g_t(z)$ given by (2.7) can be considered as a solution of the following Cauchy problem:

$$\frac{\partial g_t(z)}{\partial t} = \frac{1}{2}(\Delta_N g_t)(z); \qquad g_0(z) = g(z)$$

Putting here $g(z) = \exp \psi(z)$, $g_t(z) = \exp \psi_t(z)$, we obtain

$$\frac{\partial \psi_t(z)}{\partial t} = N(D\psi_t)(z) + 2z[(D^2\psi_t)(z) + (D\psi_t)^2(z)]; \qquad \psi_0(z) = \psi(z) \tag{2.12}$$

For every $t \geqslant 0$, the function $\psi_t(z)$ is holomorphic at zero. Thus one gets

$$\frac{\partial \psi_t^{(1)}}{\partial t} = (N+2)\, \psi_t^{(2)} + 2(\psi_t^{(1)})^2 \tag{2.13}$$

But $\psi_t^{(2)} \leqslant 0$, as follows from Lemma 2.2. Let us consider the case $\psi_t^{(2)} < 0$. Then at any point where $\psi_t^{(1)} = 0$, the right-hand side of (2.13) is strictly negative and hence the function $\psi_t^{(1)}$ is strictly decaying. For a differentiable function, such points are at most one. The case $\psi_t^{(2)} = 0$ is trivial. ∎

3. MAIN ESTIMATES

Let us return to the relation (1.11). For sufficiently small values of β, $\{f_{P_n}, n \in \mathbf{N}\}$ exists as a sequence of analytic functions. The strict meaning of this statement is now to be established by the methods derived above. For further convenience, we eliminate the explicit dependence of β by introducing the functions $f_n(z) = f_{P_n}(\beta z)$, for which we get from (1.11)

$$f_n(z) = \frac{1}{Y_n} \exp\left[\frac{1}{2}(\delta^{\lambda} - 1)\,\delta^{-n(1+\lambda)}\,\Delta_N\right][f_{n-1}(z)]^{\delta} \tag{3.1}$$

with the same Y_n. For $\chi \in \mathcal{M}_N(\lambda)$, $f_\chi(z)$ obeys (1.3) with some $\gamma_j > 0$. Denote by θ_0 the value of its parameter θ in this representation and let $\mathcal{L}_G = \{f \in \mathcal{L} \mid f(z) = \exp(\theta z)\}$, $\bar{\mathcal{L}} = \mathcal{L} \backslash \mathcal{L}_G$.

Lemma 3.1. Let $f_\chi(z)$ be chosen in $\bar{\mathcal{L}}$; then:

(i) $f_n(z) \in \bar{\mathcal{L}}$ for all $n \in \mathbf{Z}_+$ and arbitrary $\beta \in [0, 1/2\theta_0)$.

(ii) There exists an entire function $\dot{f}_n(z)$ such that for all $z \in \mathbf{C}$, $\dot{f}_n(z) = \partial f_n(z)/\partial\beta$; this statement holds for all $n \in \mathbf{Z}_+$, $\beta \in (0, 1/2\theta_0)$.

(iii) For $\theta_0 = 0$, the above statements hold for all $\beta > 0$.

Proof. For given $f(z) \in \bar{\mathcal{L}}$, denote $f(z) = \exp(\theta z)\, h(z)$, where h is the infinite product in the representation (1.3). For $t < 1/2\theta$, we get by means of (2.8) and Proposition 2.5

$$\exp\left(\frac{1}{2}\, t\, \Delta_N\right) f(z) = \exp\left(\frac{\theta z}{1 - 2t\theta}\right) h_t(z)$$

with $h_t(z)$ obeying (1.3) with $\theta = 0$. Applying this scheme in (3.1), we get

$$f_n(z) = \exp(u_n z)\, h_n(z) = \exp\left[\frac{\delta^n u_0 z}{1 - 2u_0(1 - \delta^{-n\lambda})}\right] h_n(z) \tag{3.2}$$

with $h_n(z)$ obeying (1.3) with $\theta = 0$. Here $u_0 = \beta\theta_0$ and is strictly less than 1/2 whenever $\beta \in [0, 1/2\theta_0)$. This proves statement (i). Recall that $f_0(z) = f_\chi(\beta z)$, which yields $\dot{f}_0(z) = z(Df_\chi)(\beta z) = (z/\beta)(Df_0)(z)$. This means that statement (ii) holds for $n=0$, and $\dot{f}_0(z) \in \mathscr{A}_{u_0}$. The latter is obtained by estimating $\|\dot{f}_0\|_u$ with $f_0(z)=\exp(u_0 z)\, h_0(z)$. Assume $\dot{f}_{n-1}(z) \in \mathscr{A}_{u_{n-1}}$ with u_n given by (3.2). The formal derivative can be computed from (3.1),

$$\dot{f}_n(z) = \frac{\delta}{Y_n}\left(\exp\left[\frac{1}{2}(\delta^{\lambda}-1)\,\delta^{-n(1+\lambda)}\,\Delta_N\right][f_{n-1}(z)]^{\delta-1}\dot{f}_{n-1}(z)\right.$$

$$\left. - f_n(z)\left\{\exp\left[\frac{1}{2}(\delta^{\lambda}-1)\,\delta^{-n(1+\lambda)}\,\Delta_N\right][f_{n-1}(z)]^{\delta-1}\dot{f}_{n-1}(z)\right\}_{z=0}\right)$$

As it was proven above, $f_{n-1}(z)$ obeys (3.2) hence belongs to $\mathscr{A}_{u_{n-1}}$. Thus, in view of Proposition 2.1, we have

$$\exp[\tfrac{1}{2}(\delta^{\lambda}-1)\,\delta^{-n(1+\lambda)}\,\Delta_N][f_{n-1}(z)]^{\delta-1}\dot{f}_{n-1}(z) \in \mathscr{A}_{u_n}$$

which proves statement (ii). The case $\theta_0=0$ is obvious. ∎

Denote $\varphi_n(z)=\log f_n(z)$; $\varphi_n^{(k)}=(D^k\varphi_n)(0)$; $f_n^{(k)}=(D^k f_n)(0)$, $k\in \mathbf{N}$. All these $\varphi_n^{(k)}$, $f_n^{(k)}$ are functions of β defined on $[0, 1/2\theta_0)$.

Corollary 3.1. For all $n \in \mathbf{Z}_+$, $k \in \mathbf{N}$, the functions $\varphi_n^{(k)}$ and $f_n^{(k)}$ are differentiable on $(0, 1/2\theta_0)$.

Let

$$\exp \psi_{n-1}(z) = g_{n-1}(z) = \exp(-\varphi_{n-1}^{(1)} z)\, f_{n-1}(z) \tag{3.3}$$

Insert $f_{n-1}(z)$ in the form $\exp(\varphi_{n-1}^{(1)} z)\, g_{n-1}(z)$ into (3.1) and use (2.8). Then

$$f_n(z) = \frac{1}{Y_n}(\mu_n \delta^{-1})^{N/2} \exp(\mu_n \varphi_{n-1}^{(1)} z) \exp\left(\frac{1}{2} t_n \mu_n^{-2}\Delta_N\right)[g_{n-1}(z\delta^{-2}\mu_n^2)]^{\delta} \tag{3.4}$$

where

$$\mu_n = \frac{\delta}{1-(1-\delta^{-\lambda})\,\delta^{-(n-1)(1+\lambda)}2\varphi_{n-1}^{(1)}} \tag{3.5}$$

$$t_n = \frac{\mu_n - \delta}{2\varphi_{n-1}^{(1)}} \tag{3.6}$$

For $t \in [0, t_n]$, let us consider

$$\exp R(t, z) = \exp(\tfrac{1}{2} t\, \Delta_N)[g_{n-1}(z\delta^{-2})]^{\delta} \tag{3.7}$$

If $f \in \bar{\mathscr{L}}$, then $m_1 > 0$ (see Remark 1.1) and $\varphi^{(1)} = f^{(1)} = \theta + m_1 > \theta$. Therefore, each such a function belongs to $\mathscr{L}_a$ with some $a < f^{(1)}$. Hence the function $[g_{n-1}(z\delta^{-2})]^{\delta}$ meets the conditions of Lemma 2.1 [as $g(z)$]. Then the function $\exp R(t, z)$ possesses the properties stated by Lemmas 2.1–2.3. Moreover, we have for $R(t, z)$ and $R_k(t) \equiv (D_z^k R)(t, 0)$

$$R(0, z) = \delta\psi_{n-1}(z\delta^{-2}) = -\delta^{-1}\varphi_{n-1}^{(1)} z + \delta\varphi_{n-1}(z\delta^{-2}) \tag{3.8}$$

$$R_1(0) = 0; \qquad R_k(0) = \delta^{1-2k}\varphi_{n-1}^{(k)}; \qquad k \geqslant 2 \tag{3.9}$$

Comparing (3.7) and (3.4), we also deduce

$$R(t_n, z\mu_n^2) = \varphi_n(z) - \mu_n\varphi_{n-1}^{(1)} z - \frac{N}{2}\log(\mu_n\delta^{-1}) + \log Y_n \tag{3.10}$$

$$R_1(t_n) = \mu_n^{-2}(\varphi_n^{(1)} - \mu_n\varphi_{n-1}^{(1)}) \tag{3.11}$$

$$R_k(t_n) = \mu_n^{-2k}\varphi_n^{(k)}, \qquad k \geqslant 2 \tag{3.12}$$

For $R(t, z)$, we can get an equation of the type of (2.12)

$$\frac{\partial R(t, z)}{\partial t} = N(D_z R)(t, z) + 2z[(D_z^2 R)(t, z) + (D_z R)^2(t, z)]$$

and

$$\frac{\partial R_1(t)}{\partial t} = (N+2)\, R_2(t) + 2R_1^2(t) \tag{3.13}$$

$$\frac{\partial R_2(t)}{\partial t} = (N+4)\, R_3(t) + 8R_1(t)\, R_2(t) \tag{3.14}$$

Having in mind that $R_1(0) = 0$ [see (3.9)] and applying Lemma 2.3, we obtain

$$R_1(t) < 0, \qquad \forall t \in (0, t_n] \tag{3.15}$$

By means of Lemma 2.2, we get $R_2(t) < 0$, $R_3(t) > 0$. Then taking into account (3.15) in (3.14), we have

$$\frac{\partial R_2(t)}{\partial t} > 0, \qquad t \in (0, t_n] \tag{3.16}$$

For given $t \in (0, t_n]$, there exists $\tau \in (0, t)$ such that

$$R_1(t) = t\left(\frac{\partial R_1(t)}{\partial t}\right)(\tau) = t[(N+2)\, R_2(\tau) + 2R_1^2(\tau)]$$

Taking into account (3.15) and (3.16), we obtain from the latter

$$R_1(t) > t(N+2)R_2(0), \qquad \forall t \in (0, t_n] \tag{3.17}$$

The estimates (3.15)–(3.17) and the "boundary" conditions (3.9), (3.11), and (3.12) can be used to prove the following statement.

Lemma 3.2 (Main Estimates). For all $n \in \mathbf{N}$,

$$\varphi_n^{(2)} > \delta^{-3}\mu_n^4 \varphi_{n-1}^{(2)} \tag{3.18}$$

$$\varphi_n^{(1)} < \mu_n \varphi_{n-1}^{(1)} \tag{3.19}$$

$$\varphi_n^{(1)} > \mu_n \varphi_{n-1}^{(1)} + (N+2)(1-\delta^{-\lambda})\,\mu_n^3\,\delta^{-(n-1)(1+\lambda)}\,\delta^{-3}\varphi_{n-1}^{(2)} \tag{3.20}$$

4. PROOFS AND DISCUSSION

The proofs of our theorems are based on the properties of the family $\mathscr{L}$.

Lemma 4.1. Let $\{f_n(z),\ n \in \mathbf{Z}_+ \mid f_n \in \mathscr{L}\}$ be given. Let also the derivatives $\varphi_n^{(k)} = (D^k \log f_n)(0)$ meet the following conditions:

(i) $\{\varphi_n^{(1)}\}$ converges to $\varphi \geqslant 0$.

(ii) $\{\varphi_n^{(2)}\}$ converges to zero. Then the sequence $\{f_n\}$ converges to $\exp(\varphi z)$ in $\mathscr{A}_a$ with $a = \sup\{\varphi_n^{(1)}\}$.

Proof. Let θ_n and $\gamma_j(n)$ denote the corresponding parameters f_n in the representation (1.3). Then $\theta_n \leqslant \varphi_n^{(1)}$; hence our sequence $\{f_n\}$ is bounded in $\mathscr{A}_a$, and thus its uniform convergence on compact subsets of $\mathbf{C}$ is to be shown (see Proposition 2.3). But in view of the Vitali theorem, we can show this by proving the pointwise convergence of $\{f_n\}$ on the real half-line $\mathbf{R}_+ = [0, +\infty)$. For each nonnegative α, one has $\exp(\alpha - \frac{1}{2}\alpha^2) \leqslant 1 + \alpha \leqslant \exp(\alpha)$. Making use of this, we have for arbitrary $z \in \mathbf{R}_+$

$$\exp[z(\theta_n + m_1(n)) - \tfrac{1}{2}z^2 m_2(n)] \leqslant f_n(z) \leqslant \exp[z(\theta_n + m_1(n))]$$

where, as above, $m_k(n) = \sum_j \gamma_j^k(n)$. This estimate can be rewritten as follows:

$$\exp(z\varphi_n^{(1)} - \tfrac{1}{2}z^2\varphi_n^{(2)}) \leqslant f_n(z) \leqslant \exp(z\varphi_n^{(1)})$$

The latter implies the stated convergence. ∎

To prove our theorems, we introduce [see (1.9)]

$$f_n^{(\rho)}(z) = f_n(z\delta^{-n(1+\rho)}); \qquad f_{P_n}^{(\rho)}(z) = f_{P_n}(z\delta^{-n(1+\rho)}) \tag{4.1}$$

and denote

$$\tilde{\varphi}_n(z) = \log f_n^{(0)}(z); \qquad \tilde{\varphi}_n^{(k)} = (D^k \tilde{\varphi}_n)(0) \tag{4.2}$$

$$\hat{\varphi}_n(z) = \log f_n^{(\lambda)}(z); \qquad \hat{\varphi}_n^{(k)} = (D^k \hat{\varphi}_n)(0) \tag{4.3}$$

Then we obtain $\tilde{\varphi}_n^{(k)} = \delta^{-nk} \varphi_n^{(k)}$, $\hat{\varphi}_n^{(k)} = \delta^{-n(1+\lambda)k} \varphi_n^{(k)}$. Denote also

$$\nu_n = \delta^{-1} \mu_n; \qquad \kappa_n = \delta^{-1-\lambda} \mu_n \tag{4.4}$$

where μ_n was introduced in (3.5). Then we have the following estimates as a corollary of Lemma 3.2:

$$\tilde{\varphi}_n^{(2)} > \nu_n^4 \delta^{-1} \tilde{\varphi}_{n-1}^{(2)} \tag{4.5}$$

$$\tilde{\varphi}_n^{(1)} < \nu_n \tilde{\varphi}_{n-1}^{(1)} \tag{4.6}$$

$$\tilde{\varphi}_n^{(1)} > \nu_n \tilde{\varphi}_{n-1}^{(1)} + (N+2)(1-\delta^{-\lambda})\, \delta^{-1} \nu_n^3 \delta^{-(n-1)\lambda} \tilde{\varphi}_{n-1}^{(2)} \tag{4.7}$$

$$\hat{\varphi}_n^{(2)} > \delta^{2\lambda-1} \kappa_n^4 \hat{\varphi}_{n-1}^{(2)} \tag{4.8}$$

$$\hat{\varphi}_n^{(1)} < \kappa_n \hat{\varphi}_{n-1}^{(1)} \tag{4.9}$$

$$\hat{\varphi}_n^{(1)} > \kappa_n \hat{\varphi}_{n-1}^{(1)} + (N+2)(1-\delta^{-\lambda})\, \delta^{2\lambda-1} \kappa_n^3 \hat{\varphi}_{n-1}^{(2)} \tag{4.10}$$

For given $\chi \in \mathcal{M}_N(\lambda)$, we set

$$a_\chi = -\frac{N+2}{2} \frac{\varphi_\chi^{(2)}}{(\varphi_\chi^{(1)})^2} \tag{4.11}$$

Then $a_\chi < a(\lambda)$ [see Definition 1.3 and (1.7)]; thus there exists $\varepsilon \in (0, \bar{\varepsilon})$ such that

$$a_\chi = \delta^{1/2} \frac{\delta^\varepsilon - 1}{\delta^{\lambda+\varepsilon} - 1} \tag{4.12}$$

Denote

$$\Phi^{(1)} = \frac{1}{2} \frac{\delta^\lambda - \delta^{-\varepsilon}}{\delta^\lambda - 1} = \frac{1}{2} \frac{\delta^{1/2}}{\delta^{1/2} - a_\chi} \tag{4.13}$$

$$\Phi^{(2)} = -\frac{1}{2(N+2)} \frac{(\delta^\lambda - \delta^{-\varepsilon})(1-\delta^{-\varepsilon})}{(\delta^\lambda - 1)^2} \delta^{1-\lambda-2\varepsilon} \tag{4.14}$$

The proof of Theorem 1 is based upon the following inductive lemma.

Lemma 4.2. Let the initial measure χ be chosen in $\mathcal{M}_N(\lambda)$. For this χ, let $\varepsilon, \Phi^{(1)}, \Phi^{(2)}$ be given by (4.12)–(4.14) and I_n be the triple $(i_n^1; i_n^2; i_n^3)$ of such statements:

$$i_n^1 = \{\exists \beta_n^+ \in [0, 1/2\theta_0): \hat{\varphi}_n^{(1)} = \Phi^{(1)}, \beta = \beta_n^+;\ \hat{\varphi}_n^{(1)} < \Phi^{(1)}, \beta < \beta_n^+\}$$
$$i_n^2 = \{\exists \beta_n^- \in [0, 1/2\theta_0): \hat{\varphi}_n^{(1)} = \frac{1}{2}, \beta = \beta_n^-;\ \hat{\varphi}_n^{(1)} < \tfrac{1}{2}, \beta < \beta_n^-\}$$
$$i_n^3 = \{\forall \beta \in [0, \beta_n^+]: \hat{\varphi}_n^{(2)} \geqslant \Phi^{(2)}\}$$

Then (a) I_0 is true, (b) I_n implies I_{n+1} for all $n \in \mathbf{Z}_+$.

Proof. For $n=0$, we have $\hat{\varphi}_0^{(1)} = \beta\varphi_\chi^{(1)}$, $\hat{\varphi}_0^{(2)} = \beta^2\varphi_\chi^{(2)}$. Thus we set

$$\beta_0^- = \frac{1}{2\varphi_\chi^{(1)}}; \qquad \beta_0^+ = \frac{1}{2\varphi_\chi^{(1)}} \frac{\delta^\lambda - \delta^{-\varepsilon}}{\delta^\lambda - 1} > \beta_0^- \tag{4.15}$$

and show that $\beta_0^+ < 1/2\theta_0$. For $\beta = \beta_0^+$ and a_χ given by (4.11), the definition (4.15) yields

$$\beta_0^+ \varphi_\chi^{(1)} = \hat{\varphi}_0^{(1)} = \Phi^{(1)} = \frac{1}{2} \frac{\delta^{1/2}}{\delta^{1/2} + b_N \hat{\varphi}_0^{(2)}/(\hat{\varphi}_0^{(1)})^2}; \qquad b_N = \frac{N+2}{2}$$

This equation can be solved with respect to $\hat{\varphi}_0^{(1)}$:

$$\hat{\varphi}_0^{(1)} = \frac{1}{4}\left[1 + \left(1 + 16\frac{b_N \hat{m}_2}{\delta^{1/2}}\right)^{1/2}\right]$$

where $\hat{m}_k$ are defined for $f_0^{(\lambda)}(z)$ as in Remark 1.1. Then

$$\beta_0^+ \theta_0 + \hat{m}_1 = \hat{\varphi}_0^{(1)} < \tfrac{1}{4}[1 + 1 + (16 b_N \hat{m}_2)^{1/2}] = \tfrac{1}{2} + (b_N \hat{m}_2)^{1/2} \leqslant \tfrac{1}{2} + \hat{m}_1$$

We have estimated $\hat{m}_2$ by using (1.8). This implies $\beta_0^+ < 1/2\theta_0$, hence $\beta_0^- < 1/2\theta_0$ in view of (4.15). Thus i_0^1 and i_0^2 are true. Proposition 1.1 and Definition 1.3 imply $\hat{\varphi}_0^{(2)} < 0$, $\varphi_\chi^{(2)} < 0$. For $\beta \in [0, \beta_0^+]$, this yields $\hat{\varphi}_0^{(2)} \geqslant (\beta_0^+)^2 \varphi_\chi^{(2)}$; thus

$$\begin{aligned}
\hat{\varphi}_0^{(2)} &\geqslant \frac{1}{4} \frac{\varphi_\chi^{(2)}}{(\varphi_\chi^{(1)})^2} \left(\frac{\delta^\lambda - \delta^{-\varepsilon}}{\delta^\lambda - 1}\right)^2 \\
&= -\frac{a_\chi}{2(N+2)} \left(\frac{\delta^\lambda - \delta^{-\varepsilon}}{\delta^\lambda - 1}\right)^2 \\
&= -\frac{1}{2(N+2)} \frac{(\delta^\lambda - \delta^{-\varepsilon})(1 - \delta^{-\varepsilon})}{(\delta^\lambda - 1)^2} \delta^{1/2} \geqslant \Phi^{(2)}
\end{aligned}$$

The latter estimate is based on $\frac{1}{2}+2\varepsilon\leqslant\frac{1}{2}+2\bar{\varepsilon}\leqslant 1-\lambda$. This proves i_0^3 and I_0. To prove the implication $I_{n-1}\Rightarrow I_n$, we remark that for $\beta=\beta_{n-1}^+$, i_{n-1}^1 yields $\hat{\varphi}_{n-1}^{(1)}=\Phi^{(1)}$ and $\kappa_n=\delta^{\varepsilon}$ [see (4.4) and (3.5)]. Then, by means of i_{n-1}^3, (4.14), (4.18), and (4.10), we obtain

$$\hat{\varphi}_n^{(1)}>\delta^{\varepsilon}\Phi^{(1)}+(N+2)(1-\delta^{-\lambda})\,\delta^{2\lambda-1+3\varepsilon}\Phi^{(2)}=\Phi^{(1)}$$

For $\beta=\beta_{n-1}^-$, we have $\hat{\varphi}_{n-1}^{(1)}=1/2$ and $\kappa_n=1$. Therefore, $\hat{\varphi}_n^{(1)}<1/2$ for $\beta\leqslant\beta_{n-1}^-$, as follows from i_{n-1}^2 and (4.9). Lemma 3.1 and its Corollary 3.1 imply $\hat{\varphi}_n^{(1)}$ is a continuous function of β; thus there exists at least one value of $\beta=\tilde{\beta}_n^+\in(\beta_{n-1}^-,\beta_{n-1}^+)$ such that $\hat{\varphi}_n^{(1)}=\Phi^{(1)}$. The smallest such one is set to be β_n^+. The existence of $\beta_n^-\in(\beta_{n-1}^-,\beta_n^+)$ can be established in the same way. For $\beta\leqslant\beta_n^+$, we have $\beta\leqslant\beta_{n-1}^+$ and then $\hat{\varphi}_{n-1}^{(1)}\leqslant\Phi^{(1)}$ due to i_{n-1}^1. This yields $\kappa_n\leqslant\delta^{\varepsilon}$. Then we get from (4.8)

$$\hat{\varphi}_n^{(2)}>\delta^{2\lambda-1+4\varepsilon}\hat{\varphi}_{n-1}^{(2)}\geqslant\hat{\varphi}_{n-1}^{(2)}\geqslant\Phi^{(2)}$$

where the following estimates were used: $\hat{\varphi}_n^{(2)}<0$, $\forall n\in\mathbf{Z}_+$; $2\lambda-1+4\varepsilon<0$. ▌

Let us consider the set $\Delta_n=\{\beta\in\mathbf{R}_+\mid\frac{1}{2}<\hat{\varphi}_n^{(1)}<\Phi^{(1)}\}$. Lemma 4.2 implies $\Delta_n\subseteq(\beta_n^-,\beta_n^+)$, Δ_n is nonempty and open. Let us prove $\Delta_n\subseteq\Delta_{n-1}$. Suppose there exists some $\beta\in\Delta_n$ that does not belong to Δ_{n-1}. For this β, either $\hat{\varphi}_{n-1}^{(1)}\leqslant 1/2$ or $\hat{\varphi}_{n-1}^{(1)}\geqslant\Phi^{(1)}$. Hence, either $\hat{\varphi}_n^{(1)}<1/2$ or $\hat{\varphi}_n^{(1)}>\Phi^{(1)}$ in view of arguments used above. This runs counter to the supposition $\beta\in\Delta_n$. Let D_n be the closure of Δ_n; then $D_n\subseteq D_{n-1}$, D_n is nonempty, $D_n\subseteq\{\beta\mid\frac{1}{2}\leqslant\hat{\varphi}_n^{(1)}\leqslant\Phi^{(1)}\}$. Let $D_*=\bigcap_n D_n$; then D_* is closed and nonempty. Denote $\beta_*=\inf D_*=\min D_*$. Then:

(i) For $\beta=\beta_*$,

$$\tfrac{1}{2}<\hat{\varphi}_n^{(1)}<\Phi^{(1)},\qquad\forall n\in\mathbf{Z}_+\tag{4.16}$$

(ii) For $\beta<\beta_*$, there exists $n(\beta)$ such that $\hat{\varphi}_{n(\beta)}^{(1)}<1/2$.

Indeed, from the definition, $D_*\subseteq\{\beta\mid\frac{1}{2}\leqslant\hat{\varphi}_n^{(1)}\leqslant\Phi^{(1)}\}$, $\forall n\in\mathbf{Z}_+$. Suppose $\hat{\varphi}_n^{(1)}=1/2$ for some n; then $\hat{\varphi}_m^{(1)}<1/2$ for all $m>n$, which may occur whenever β_* does not belong to all D_m with $m>n$. The latter contradicts the definition of β_*. The case $\hat{\varphi}_n^{(1)}=\Phi^{(1)}$ can be excluded similarly. Suppose there exists some $\beta<\beta_*$ such that (4.16) holds. Then this β belongs to D_*, which contradicts the definition of β_*. The case $\hat{\varphi}_n^{(1)}\geqslant\Phi^{(1)}$ is impossible for this β, because $\beta_*\leqslant\inf\{\beta_n^+\}$, which yields $\hat{\varphi}_n^{(1)}<\Phi^{(1)}$ in view of i_n^1. In what follows, we prove such a statement.

Lemma 4.3. Let the conditions of Lemma 4.2 be fulfilled; then there exists $\beta_*>0$ such that the inequalities (4.16) hold. For $\beta<\beta_*$, there exists $n(\beta)$ such that $\hat{\varphi}_{n(\beta)}^{(1)}<1/2$.

Proof of Theorem 1. Let the initial measure χ be chosen in $\mathscr{M}_N(\lambda)$. For this χ, there exists ε obeying (4.12). Then Lemmas 4.2 and 4.3 may be used. For $\beta=\beta_*$, (4.16) holds, yielding $\kappa_n<\delta^{\varepsilon}$. Applying this estimate in (4.8), we get

$$|\hat{\varphi}_n^{(2)}|<\delta^{(2\lambda-1+4\varepsilon)n}\,|\hat{\varphi}_0^{(2)}|=\delta^{(2\lambda-1+4\varepsilon)n}\,|\Phi^{(2)}|$$

In view of $\varepsilon<\bar{\varepsilon}\leqslant(1-2\lambda)/4$, this gives

$$\hat{\varphi}_n^{(2)}\to 0,\qquad n\to+\infty \tag{4.17}$$

Let us prove $\hat{\varphi}_n^{(1)}\to 1/2$ for $\beta=\beta_*$. For this aim, rewrite (4.10) as follows:

$$(2\hat{\varphi}_n^{(1)}-1)>\delta^{\lambda}(2\hat{\varphi}_{n-1}^{(1)}-1)+\delta^{\lambda}(\delta^{\lambda}-1)\,\kappa_n\Psi_n \tag{4.18}$$

$$\Psi_n=(2\hat{\varphi}_{n-1}^{(1)}-1)^2+\frac{2(N+2)}{\delta}\kappa_n^2\hat{\varphi}_{n-1}^{(2)} \tag{4.19}$$

Show that $\Psi_n\leqslant 0$ for all $n\in\mathbf{Z}_+$. If $\Psi_m>0$, then (4.18) yields

$$(2\hat{\varphi}_m^{(1)}-1)>\delta^{\lambda}(2\hat{\varphi}_{m-1}^{(1)}-1)$$

The latter implies

$$\begin{aligned}\Psi_{m+1}&=(2\hat{\varphi}_m^{(1)}-1)^2+\frac{2(N+2)}{\delta}\kappa_{m+1}^2\hat{\varphi}_m^{(2)}\\&>\delta^{2\lambda}(2\hat{\varphi}_{m-1}^{(1)}-1)^2+\frac{2(N+2)}{\delta}\delta^{2\varepsilon}\kappa_m^2\,\delta^{2\lambda-1+4\varepsilon}\hat{\varphi}_{m-1}^{(2)}\\&=\delta^{2\lambda}\left[(2\hat{\varphi}_{m-1}^{(1)}-1)^2+\frac{2(N+2)}{\delta}\delta^{6\varepsilon-1}\kappa_m^2\hat{\varphi}_{m-1}^{(2)}\right]>\delta^{2\lambda}\Psi_m\end{aligned} \tag{4.20}$$

Here we have used $\varepsilon<\bar{\varepsilon}\leqslant 1/6$; $1<\kappa_m<\delta^{\varepsilon}$, if $\hat{\varphi}_{m-1}^{(1)}<\Phi^{(1)}$, and $\kappa_{m+1}<\delta^{\varepsilon}\kappa_m$. Thus, assuming $\Psi_m>0$, we have (4.20) and then

$$\hat{\varphi}_{n+m}^{(1)}>\tfrac{1}{2}+\delta^{(n+1)\lambda}(\hat{\varphi}_{m-1}^{(1)}-\tfrac{1}{2}),\qquad n\in\mathbf{Z}_+$$

which contradicts the boundedness of $\{\hat{\varphi}_n^{(1)}\}$. Thus $\Psi_n\leqslant 0$ and (4.19) gives

$$(2\hat{\varphi}_n^{(1)}-1)^2\leqslant 2(N+2)\,\delta^{1+2\varepsilon}\,|\hat{\varphi}_n^{(2)}|$$

Taking into account (4.17), we obtain the stated convergence of $\{\hat{\varphi}_n^{(1)}\}$. To complete the proof, we use Lemma 4.1 and obtain that the sequence $\{f_n^{(\lambda)},\,n\in\mathbf{Z}_+\}$ convergence to $\exp(\frac{1}{2}z)$, which gives in turn for the sequence $\{f_{P_n}^{(\lambda)},\,n\in\mathbf{Z}_+\}$ to converge to $\exp(z/2\beta_*)$. ▮

To prove the second theorem, we use certain additional properties of our model. Begin with the following assertion.

Lemma 4.4. For every $\beta<\beta_*$, there exists $\zeta\in(0,\lambda)$ such that for all $n\geqslant n(\beta)$ (defined in Lemma 4.3), the following estimate holds:

$$\hat{\varphi}_n^{(1)}<\tfrac{1}{2}\delta^{-(n-n(\beta))\zeta} \tag{4.21}$$

Proof. Lemma 4.3 yields that (4.21) holds for $n=n(\beta)$. In this case $\kappa_{n+1}<1$ and (4.9) gives $\hat{\varphi}_{n+1}^{(1)}<\hat{\varphi}_n^{(1)}$. Denote

$$\zeta=\min\left\{\log\frac{\hat{\varphi}_{n(\beta)}^{(1)}}{\hat{\varphi}_{n(\beta)+1}^{(1)}};\qquad \log(\delta^{\lambda}-1)\right\}\Big/\log\delta \tag{4.22}$$

Then (4.21) holds for $n(\beta)+1$. Let it hold for some $n-1$. Then (4.9) yields

$$\hat{\varphi}_n^{(1)}<\frac{1}{2}\delta^{-(n-n(\beta))\zeta}\frac{\delta^{\zeta-\lambda}}{1-(1-\delta^{-\lambda})\,\delta^{-(n-n(\beta)-1)\zeta}}$$

But from the definition (4.22), one has $\delta^{\zeta}<\delta^{\lambda}-1$, which gives

$$\frac{\delta^{\zeta-\lambda}}{1-(1-\delta^{-\lambda})\,\delta^{-(n-n(\beta)-1)\zeta}}<1$$

for all $n>n(\beta)+1$. Therefore, (4.21) holds for all $n\geqslant n(\beta)$. ▮

The estimate (4.21) implies that the product $\prod_{n=n(\beta)}^{\infty}v_n$ converges.

Corollary 4.1. For $\beta<\beta_*$, there exists $C>0$ such, that for all $n>n(\beta)$,

$$v_n v_{n-1}\cdots v_{n(\beta)}\leqslant C \tag{4.23}$$

$$|\tilde{\varphi}_n^{(2)}|<\delta^{-(n-n(\beta))}C^4\,|\tilde{\varphi}_{n(\beta)}^{(2)}| \tag{4.24}$$

The estimate (4.24) follows from (4.23) and (4.5). It yields $\tilde{\varphi}_n^{(2)}\to 0$; thus, to prove Theorem 2, we have only to show the convergence of $\{\tilde{\varphi}_n^{(1)}\}$.

Lemma 4.5. For $\beta<\beta_*$, the sequence $\{\tilde{\varphi}_n^{(1)}\}$ is fundamental and hence convergent.

Proof. For $\beta<\beta_*$, we choose $n>p>n(\beta)$, and show that $|\tilde{\varphi}_n^{(1)}-\tilde{\varphi}_p^{(1)}|$ can be made arbitrarily small. For this aim, we use the estimates (4.6) and (4.7). The first one gives

$$\tilde{\varphi}_n^{(1)}<v_n v_{n-1}\cdots v_{p+1}\tilde{\varphi}_p^{(1)}$$

while (4.7) yields in turn

$$\tilde{\varphi}_n^{(1)} > \nu_n \nu_{n-1} \cdots \nu_{p+1} \tilde{\varphi}_p^{(1)} + (N+2)\,\delta^{-1}(1-\delta^{-\lambda}) \times \sum_{s=p}^{n-1} \nu_n \nu_{n-1} \cdots \nu_{s+2} \nu_{s+1}^3\, \delta^{-\lambda s} \tilde{\varphi}_s^{(2)}$$

Taking into account that $\nu_s > 1$, $\tilde{\varphi}_s^{(2)} < 0$, and the estimates (4.23) and (4.24), we get

$$\begin{aligned}\tilde{\varphi}_n^{(1)} &> \nu_n \nu_{n-1} \cdots \nu_{p+1} \tilde{\varphi}_p^{(1)} + (N+2)(1-\delta^{-\lambda})\,\delta^{-1} C^3 \sum_{s=p}^{n-1} \delta^{-\lambda s} \tilde{\varphi}_s^{(2)} \\ &> \nu_n \nu_{n-1} \cdots \nu_{p+1} \tilde{\varphi}_p^{(1)} - (N+2)(1-\delta^{-\lambda})\,\delta^{-1} C^7 \delta^{n(\beta)} \frac{\delta^{-p(1+\lambda)}}{1-\delta^{-1-\lambda}}\end{aligned}$$

Therefore,

$$(\nu_n \nu_{n-1} \cdots \nu_{p+1} - 1)\, \tilde{\varphi}_p^{(1)} > \tilde{\varphi}_n^{(1)} - \tilde{\varphi}_p^{(1)} > (\nu_n \nu_{n-1} \cdots \nu_{p+1} - 1)\, \tilde{\varphi}_p^{(1)} - A\delta^{-p(1+\lambda)}$$

where A does not depend on p and n. Let us estimate the product of ν. Keeping in mind (4.21), we get

$$\begin{aligned}\nu_n \nu_{n-1} \cdots \nu_{p+1} &< \prod_{s=0}^{\infty} [1 - a\delta^{-s\zeta}]^{-1} = \exp\left(\sum_{l=1}^{\infty} \frac{a^l}{l} \frac{1}{1-\delta^{-l\zeta}}\right) \\ &\leqslant \exp\left(\frac{1}{1-\delta^{-\zeta}} \sum_{l=1}^{\infty} \frac{a^l}{l}\right) = (1-a)^{-b}\end{aligned}$$

where

$$a = (1-\delta^{-\lambda})\,\delta^{-(p-n(\beta))\zeta}; \qquad b = (1-\delta^{-\zeta})^{-1}$$

Then

$$\begin{aligned}0 < \nu_n \nu_{n-1} \cdots \nu_{p+1} - 1 &< (1-a)^{-b} - 1 < \frac{ba}{(1-a)^b} \\ &< (1-\delta^{-\lambda})\, \delta^{n(\beta)\zeta + \lambda b} b \delta^{-p\zeta}\end{aligned}$$

The latter gives

$$K_0\, \delta^{-p\zeta} < \tilde{\varphi}_n^{(1)} - \tilde{\varphi}_p^{(1)} < K_1\, \delta^{-p\zeta}$$

with appropriate $K_i > 0$. ∎

The proof of Theorem 2 follows directly from Lemmas 4.1, 4.4, and 4.5. We conclude the paper with the following remarks.

1. For $\lambda \in (0, 1/2)$, the behavior of our hierarchical model at and above the critical temperature is similar for all initial measures χ which are chosen in the Lee–Yang class of measures and satisfy the restrictions (1.8). This behavior corresponds to that described by Bleher and Major.[2, 3]

2. The first restriction in (1.8) seems to be purely technical. But it actually plays a considerably more important role in our theory. This restriction selects the starting measures for which the exponential decay of the sequence $\{|\hat{\varphi}_n^{(2)}|\}$ [see (4.17) and above] holds $\forall n \in \mathbf{Z}_+$. We believe, but unfortunately cannot prove, that for χ with large $|\varphi_\chi^{(2)}|$, the decay of $\{|\hat{\varphi}_n^{(2)}|\}$ will be slower until i_n^3 becomes valid. This slow decay will last longer and longer (with respect to n) when $\Phi^{(2)}$ becomes smaller and smaller with λ approaching 1/2. Such a slow decay for all $n \in \mathbf{Z}_+$ will characterize the situation at the critical point and $\lambda = 1/2$. This picture agrees with that described in refs. 4 and 6. Such an agreement has another supporting argument. It lies in the fact that the triviality of the φ_4^4 model (corresponding to $\lambda = 1/2$) is also based on the second Lebowitz inequality [see Eq. (15.54) in ref. 4]. But the latter was proved to hold for all ferromagnetic spin models with the initial measure χ possessing the Lee–Yang property.[10]

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